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ON SOME RESULTS OF HYPERGEOMETRIC BERNOULLI NUMBERS AND POLYNOMIALS (PARTLY JOINT WORK WITH TAKAO KOMATSU AND MIN-SOO KIM)

SU HU

ABSTRACT. In this note, we shall give a survey on our recent results concerning hypergeometric Bernoulli numbers and polynomials, including sums of products identity, differential equations, recurrence relation, closed form and determinant expressions.

1. INTRODUCTION

The Bernoulli polynomials $B_n(x)$ are defined by the following generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{1.1}$$

and $B_n=B_n(0)$ are named Bernoulli numbers. These numbers and polynomials have a long history, which arise from Bernoulli's calculations of power sums in 1713, that is,

$$\sum_{j=1}^{m}j^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1}$$

(see [41, p.5, (2.2)]). They have many applications in modern number theory, such as modular forms [25] and Iwasawa theory [24]. A recent book by Arakawa, Ibukiyama and Kaneko [1] give a nice introduction of Bernoulli numbers and polynomials including their connections with zeta functions.

For $r\in\mathbb{N}$, in 1924, Nörlund [33] generalized (1.1) to give a definition of higher order Bernoulli polynomials and numbers

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.2}$$

We also have a similar expression of multiple power sums

$$\sum_{l_1,\dots,l_n=0}^{m-1}(t+l_1+\cdots+z_n)^k$$

in terms of higher order Bernoulli polynomials (see [28, Lemma 2.1]).

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For $N \in \mathbb{N}$, in 1967, Howard [15, 16] modified the generating function of Bernoulli polynomials to define a new polynomial sequences $B_{N,n}(x)$ as follows

$$\frac{t^N e^{xt}/N!}{e^t - T_{N-1}(t)} = \sum_{n=0}^{\infty} B_{N,n}(x)\frac{t^n}{n!}, \tag{1.3}$$

where $T_{N-1}(t)$ is the Taylor polynomial of order $N-1$ for the exponential function. If $N=1$, this reduces to the classical Bernoulli polynomials, that is, $B_{1,n}(x) = B_n(x)$.

Let $(a)_n$ be the Pochhammer symbol

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & (n \geq 1) \\ 1 & (n = 0) \end{cases}.$$

The confluent hypergeometric function ${}_1F_1(a;b;t)$ is defined by

$${}_1F_1(a;b;t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{t^n}{n!}$$

(see [26, p. 2261]). Because the generating function $f(t) = \frac{e^t - T_{N-1}(t)}{t^N/N!}$ can be expressed as ${}_1F_1(1; N+1; t)$, that is,

$$\frac{t^N e^{xt}/N!}{e^t - T_{N-1}(t)} = \frac{e^{xt}}{{}_1F_1(1; N+1; t)} = \sum_{n=0}^{\infty} B_{N,n}(x)\frac{t^n}{n!}, \tag{1.4}$$

it is reasonable to name the polynomials $B_{N,n}(x)$ hypergeometric Bernoulli polynomials and the numbers $B_{N,n} = B_{N,n}(0)$ hypergeometric Bernoulli numbers .

2. Results

In this section, we shall introduce our recent results with Takao Komatsu and Min-Soo Kim on hypergeometric Bernoulli numbers and polynomials, including sums of products identity, differential equations, recurrence relation, closed form and determinant expressions. All the proofs can be found in several papers listed in the references.

The classical Bernoulli numbers and polynomials satisfy many interesting identities. The most remarkable one is Euler's sums of products identity

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \geq 1). \tag{2.1}$$

This identity has been generalized by many authors from different directions (see [3, 8, 11, 27, 29, 37, 38]). In particular, Dilcher [11] provided explicit expressions for sums of products for arbitrarily many Bernoulli numbers and polynomials, and Hu and Kim [29] obtained the sums of products identity for the Apostol-Bernoulli numbers by expressing them in terms of the special values of multiple Hurwitz-Lerch zeta functions at non-positive integers.

As the Bernoulli numbers and polynomials, the hypergeometric Bernoulli numbers and polynomials also satisfy many interesting properties ([7, 10, 18, 19, 32, 34, 35, 36]).

In 2010, Kamano [26] proved the following result for sums of products of hypergeometric Bernoulli numbers, which is a generalization of Euler's and Dilcher's works (see [11]).

Theorem 2.1 (Kamano [26, p. 2262, Main Theorem]). *Let N and r be positive integers. For any integer $n \geq r-1$, we have*

$$\sum_{\substack{i_1,\dots,i_r\geq 0\\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\cdots i_r!} B_{N,i_1}\cdots B_{N,i_r}$$
$$= \frac{1}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}(i;1+N(r-1)-n)(-1)^i\binom{n}{i} i! B_{N,n-i},$$

where $A_r^{(N)}(i;s)\in\mathbb{Q}[s](0\leq i\leq r-1)$ are polynomials defined by the following recurrence relation:

$$\begin{aligned} &A_1^{(N)}(0;s)=1\\ &A_r^{(N)}(i;s)=\frac{s-1}{r-1}A_{r-1}^{(N)}(i;s-N)+A_{r-1}^{(N)}(i-1;s-N+1). \end{aligned} \tag{2.2}$$

Here $r\geq 2$ and $A_r^{(N)}(i;s)$ are defined to be zero for $i\leq -1$ and $i\geq r$.

Recently, Hu and Kim [21] obtained the following sums of products of hypergeometric Bernoulli numbers which generalized the above results.

Theorem 2.2 (Hu and Kim [21, Theorem 1.2]). *Let N and r be positive integers and let $x=x_1+\dots+x_r$. For any integer $n\geq r-1$, we have*

$$\sum_{\substack{i_1,\dots,i_r\geq 0\\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\cdots i_r!} B_{N,i_1}(x_1)\cdots B_{N,i_r}(x_r)$$
$$= \frac{1}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}(i,x;1+N(r-1)-n)(-1)^i\binom{n}{i} i! B_{N,n-i}(x),$$

where $A_r^{(N)}(i,x;s)\in\mathbb{Q}[x,s](0\leq i\leq r-1)$ are polynomials defined by the recurrence relation:

(2.3)

$$\begin{aligned} &A_1^{(N)}(0,x;s)=1\\ &A_r^{(N)}(i,x;s)=\frac{s-1}{r-1}A_{r-1}^{(N)}(i,x;s-N)-\frac{x-(r-1)}{r-1}A_{r-1}^{(N)}(i-1,x;s-N+1). \end{aligned}$$

Here $r\geq 2$ and $A_r^{(N)}(i,x;s)$ are defined to be zero for $i\leq -1$ and $i\geq r$.

For $N, r \in \mathbb{N}$, according to Nörlund (1.2), we may also define the higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ by the generating function

$$\left(\frac{t^N/N!}{e^t - T_{N-1}(t)}\right)^r e^{xt} = \frac{e^{xt}}{({}_1F_1(1; N+1; t))^r} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x)\frac{t^n}{n!} \tag{2.4}$$

(see [21, (1.18)]). The higher order hypergeometric Bernoulli numbers are defined by $B_{N,n}^{(r)} = B_{N,n}^{(r)}(0)$ (see [26, 36]). When $r = 1$, we obtain the hypergeometric Bernoulli polynomials $B_{N,n}(x) = B_{N,n}^{(1)}(x)$ and $B_{N,n} = B_{N,n}^{(1)}(0)$ is the hypergeometric Bernoulli numbers.

Using the properties of Appell polynomials [2, 40], Hu and Kim showed that the higher order hypergeometric Bernoulli polynomials satisfy the following differential equation.

Theorem 2.3 (Hu and Kim [21, Theorem 1.5]). *The higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ satisfy the differential equation*

$$\frac{B_{N,n}}{n!}y^{(n)} + \frac{B_{N,n-1}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_{N,2}}{2!}y'' - \left(\frac{x}{rN} - \frac{1}{N(N+1)}\right)y' + \frac{n}{rN}y = 0.$$

Hu and Kim also obtained a linear recurrence for higher order hypergeometric Bernoulli polynomials which generalized the results of Lu [32] and He-Ricci [17].

Theorem 2.4 (Hu and Kim [21, Theorem 1.8]). *For $n \in \mathbb{N}$, the higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ satisfy the recurrence*

$$B_{N,n+1}^{(r)}(x) = \left(x - \frac{r}{N+1}\right)B_{N,n}^{(r)}(x) - rN\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{N,n-k+1}}{n-k+1}B_{N,k}^{(r)}(x).$$

The special case $N = 1$ gives the following statement.

Corollary 2.5 (Lu [32, Theorem 2.1]). *For $n \in \mathbb{N}$, the higher order Bernoulli polynomials $B_n^{(r)}(x)$ satisfy the recurrence*

$$B_{n+1}^{(r)}(x) = \left(x - \frac{1}{2}r\right)B_n^{(r)}(x) - r\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{n-k+1}(1)}{n-k+1}B_k^{(r)}(x).$$

Let $N = r = 1$ and replace n by $n+1$ in Theorem 2.4 to obtain the next result.

Corollary 2.6 (He and Ricci [17, Theorem 2.2]). *For $n \in \mathbb{N}$, the Bernoulli polynomials $B_n(x)$ satisfy the recurrence*

$$B_n(x) = \left(x - \frac{1}{2}\right)B_{n-1}(x) - \frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}B_{n-k}B_k(x).$$

Remark 2.7. The special case $r = 1$ gives some results presented in [34].

Hu and Komatsu [23] further defined the higher order generalized hypergeometric Bernoulli polynomials $B_{M,N,n}^{(r)}(x)$, that is,

$$\frac{e^{xt}}{\left({}_1F_1(M;M+N;t)\right)^r}=\sum_{n=0}^{\infty}B_{M,N,n}^{(r)}(x)\frac{t^n}{n!}\,. \tag{2.5}$$

When $x=0$, $B_{M,N,n}^{(r)}=B_{M,N,n}^{(r)}(0)$ are the higher order generalized hypergeometric Bernoulli numbers. When $M=1$, we have the higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)=B_{1,N,n}^{(r)}(x)$ (see (2.4) above).

By using the Hasse-Teichmüller derivatives, Hu and Komatsu [23] obtained the following closed form expression of the higher order generalized hypergeometric Bernoulli numbers $B_{M,N,n}^{(r)}$. According to Wiki (also see Qi and Chapman [39]), "In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit."

Theorem 2.8 (Hu and Komatsu [23, Theorem 1]). *For $M,N,n\ge 1$, we have*

$$B_{M,N,n}^{(r)}=n!\sum_{k=1}^{n}(-1)^k\sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}}D_r(e_1)\cdots D_r(e_k)\,,$$

where

$$D_r(e)=\sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}}\frac{(M)^{(i_1)}}{(M+N)^{(i_1)}i_1!}\cdots\frac{(M)^{(i_r)}}{(M+N)^{(i_r)}i_r!}. \tag{2.6}$$

Hu and Komatsu [23] also obtained a determinant expression of $B_{M,N,n}^{(r)}$.

Theorem 2.9 (Hu and Komatsu [23, Theorem 2]). *For $M,N,n\ge 1$, we have*

$$B_{M,N,n}^{(r)}=(-1)^n n!\begin{vmatrix} D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-2) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-1) & \cdots & D_r(2) & D_r(1) \end{vmatrix}.$$

where $D_r(e)$ are given in (2.6).

When $r=1$, we have the following determinant of the generalized hypergeometric Bernoulli numbers $B_{M,N,n}$.

Theorem 2.10 (Hu and Komatsu [23, Theorem 3]). *For $n \geq 1$, we have*

$$B_{M,N,n} = (-1)^n n! \begin{vmatrix} \frac{(M)^{(1)}}{(M+N)^{(1)}} & 1 & & & \\ \frac{(M)^{(2)}}{2!(M+N)^{(2)}} & \frac{(M)^{(1)}}{(M+N)^{(1)}} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(M)^{(n-1)}}{(n-1)!(M+N)^{(n-1)}} & \frac{(M)^{(n-2)}}{(n-2)!(M+N)^{(n-2)}} & \cdots & \frac{(M)^{(1)}}{(M+N)^{(1)}} & 1 \\ \frac{(M)^{(n)}}{n!(M+N)^{(n)}} & \frac{(M)^{(n-1)}}{(n-1)!(M+N)^{(n-1)}} & \cdots & \frac{(M)^{(2)}}{2!(M+N)^{(2)}} & \frac{(M)^{(1)}}{(M+N)^{(1)}} \end{vmatrix}.$$

Remark 2.11. In fact, Theorems 2.8, 2.9 and 2.10 above are stated in a more general situation in Hu and Komatsu's work [23], in fact, they are all established for the related numbers of Appell polynomials (see [2, 6] for the definitions of Appell polynomials (of higher order)).

Letting $M = N = 1$ in the above theorem, we obtain the following classical determinant expression of Bernoulli numbers which was stated in an article by Glaisher [12] in 1875:

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}. \tag{2.7}$$

Remark 2.12. Recently, applying the Hasse-Teichmüller derivatives [13], Komatsu and Yuan [30] also presented a determinant expression of the higher order generalized hypergeometric Cauchy numbers (see [30, Theorem 4]). When $r = M = N = 1$, their formula recovers the classical determinant expression of Cauchy numbers c_n which was also stated in the article of Glaisher [12, p.50]:

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix}.$$

From the integral expression for the generating function of Bernoulli polynomials, Qi and Chapman [39] got new closed form and determinant expressions of Bernoulli polynomials, which reduces to another determinant expression of Bernoulli numbers. In [22], by directly applying the generating functions instead of their integral expressions, Hu and Kim obtained new closed form and determinant expressions of Apostol-Bernoulli polynomials [4, 5, 31].

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Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
E-mail address: mahusu@scut.edu.cn